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AN ALTERNATIVE GENERAL SOLUTION OF THE STEADY-STATE HEAT DIFFUSION EQUATION

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Abstract – A general solution is obtained for the steady-state heat diffusion equation by the application of the finite integral transform technique. The present solution contains one less infinite series than that obtainable by the integral transform technique in which all partial derivatives with respect to the space variables are removed from the differential equation by integral transformation. Several special cases are readily obtainable from the present solution. The application of the general result to the solution of specific problems is illustrated with examples.

NOMENCLATURE

B,	boundary condition operator defined by equation (1f);
$\frac{\partial}{\partial n}$,	normal derivative in the outward direction at the boundary surface;
$f_k(\mathbf{x})$,	source function at $t = t_k$, defined by equation (1c, d);
L,	differential operator in the space variables $\mathbf{x}$, defined by equation (1e);
L_t,	a second order linear differential operator in the variable t;
S,	boundary surface of the region V;
$T(\mathbf{x}, t)$,	temperature;
t,	the space variable that is not to be transformed;
$\mathbf{x}$,	the space variables that are to be transformed;
$w(\mathbf{x})$,	a prescribed function in equation (1a);
$\alpha(\mathbf{x}), \beta(\mathbf{x})$,	boundary condition coefficients in equation (1f);
δ_k, γ_k,	coefficients in equations (1c, d);
μ_i,	eigenvalue;
$\psi(\mu_i, \mathbf{x})$,	$\equiv \psi_i(\mathbf{x})$ eigenfunction.

INTRODUCTION

THE INTEGRAL transform technique has been used for the solution of time dependent and steady-state heat diffusion equation in several references[1–3]. In the case of steady-state problems, if all the partial derivatives with respect to the space variables are removed from the system by the application of finite integral transform technique and the transform of the function obtained in this manner is inverted, the resulting solution contains triple summation for a three dimensional problem and double summation for a two dimensional problem. In an alternative approach, the system can be reduced to an ordinary differential equation in one of the space variables by removing from the system all the space variables except one by integral transformation, the resulting ordinary differential equation is solved and the transform is inverted. The solution obtained with the second method contains one less summation than that obtained by the first approach. In the following analysis, we present a general solution of the steady-state heat diffusion equation for finite regions by the latter approach.

ANALYSIS

Let t be the space variable that will not be transformed in the integral transformation process and L_t an arbitrary, second order linear differential operator associated with the space variable t. Let $\mathbf{x}$ denote the remaining space variables that will be transformed by the application of the integral transform and L the differential operator associated with the space variables $\mathbf{x}$. Then, with this formalism, we write steady-state heat conduction problem for a finite region V as

$$\{w(\mathbf{x})L_t + L\}T(\mathbf{x}, t) = 0, \quad \mathbf{x} \in V, \; t_0 \le t \le t_1 \tag{1a}$$

$$BT(\mathbf{x}, t) = 0, \quad \mathbf{x} \in S \tag{1b}$$

$$\left\{\delta_k - (-1)^k \gamma_k \frac{\partial}{\partial t}\right\} T(\mathbf{x}, t_k) = f_k(\mathbf{x}), \; k = 0, 1, \tag{1c, d}$$

where

$$L \equiv -\nabla \cdot [k(\mathbf{x})\nabla] + d(\mathbf{x}) \tag{1e}$$

$$B \equiv \alpha(\mathbf{x}) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial}{\partial n} \tag{1f}$$

and L_t is an arbitrary second order differential operator in the space variable t. We assume further that $\alpha(\mathbf{x})$ does not vanish simultaneously at all points $\mathbf{x} \in S$.

Appropriate eigenvalue problem for the solution of this system is taken as

$$L\psi(\mathbf{x}) = \mu^2 w(\mathbf{x})\psi(\mathbf{x}), \quad \mathbf{x} \in V \tag{2a}$$

$$B\psi(\mathbf{x}) = 0, \quad \mathbf{x} \in S \tag{2b}$$

and the eigenfunctions $\psi(\mu_i, \mathbf{x}) \equiv \psi_i(\mathbf{x})$ and the eigenvalues μ_i of this eigenvalue problem are considered to be known.

To solve the problem (1) by the integral transform technique, the finite integral transform pair is defined as [2]: the integral transform

$$\bar{T}_i(t) = \int_V w(\mathbf{x})\psi_i(\mathbf{x})T(\mathbf{x}, t)\mathrm{d}v \equiv (\psi_i, T) \tag{3a}$$

and the inversion formula

$$T(\mathbf{x}, t) = \sum_{i=1}^{\infty} \frac{1}{N_i} \psi_i(\mathbf{x})\bar{T}_i(t), \tag{3b}$$

where N_i is the normalization integral given by

$$N_i = \int_V w(\mathbf{x})[\psi_i(\mathbf{x})]^2 \mathrm{d}x \equiv (\psi_i, \psi_i). \tag{3c}$$

We take the integral transform of system (1) by the application of transform (3a). That is, equation (1a) is multiplied by $\psi_i(\mathbf{x})$, equation (2a) by $T(\mathbf{x}, t)$, the results are subtracted, integrated over the region V, the definition of the integral transform (3a) is utilized, the volume integral is transformed to the surface integral by Green's theorem, and the boundary conditions (1b) and (2b) applied. We obtain the following second order ordinary differential equation for the transform $\bar{T}_i(t)$:

$$\{L_t + \mu_i^2\}\bar{T}_i(t) = 0 \quad \text{in } t_0 \le t \le t_1 \tag{4a}$$

$$\left\{\delta_k - (-1)^k \gamma_k \frac{\mathrm{d}}{\mathrm{d}t}\right\} \bar{T}_i(t) = (\psi_i, f_k), \quad k = 0, 1. \tag{4b, c}$$

Let $u(\mu_i, t)$ and $v(\mu_i, t)$ be two linearly independent solutions of the differential equation (4a). The general solution for $\bar{T}_i(t)$ is constructed by taking a linear combination of these two solutions as

$$\bar{T}_i(t) = Cu(\mu_i, t) + Dv(\mu_i, t). \tag{5}$$

If the solution (5) should satisfy the two boundary conditions (4b, c), we have

$$CU(\mu_i, t_k) + DV(\mu_i, t_k) = (\psi_i, f_k), \quad k = 0, 1 \tag{6a, b}$$

where

$$U(\mu_i, t_k) = \left\{\delta_k - (-1)^k \gamma_k \frac{\mathrm{d}}{\mathrm{d}t}\right\} u(\mu_i, t_k) \tag{7a}$$

$$V(\psi_i, t_k) = \left\{\delta_k - (-1)^k \gamma_k \frac{\mathrm{d}}{\mathrm{d}t}\right\} v(\mu_i, t_k). \tag{7b}$$

When equations (6a, b) are solved for C and D, the results are introduced into equation (5) and the resulting integral transform $\bar{T}_i(t)$ is inverted by the inversion formula (3b), the solution for $T_i(t)$ is inverted by the inversion formula (3b), the solution for $T(\mathbf{x}, t)$ is obtained as

$$T(\mathbf{x}, t) = \sum_{i=1}^{\infty} \frac{\psi_i(\mathbf{x})}{(\psi_i, \psi_i)} \frac{\sum_{k=0}^{1} (-1)^{1-k}(\psi_i, f_{1-k})[V(\mu_i, t_k)u(\mu_i, t) - U(\mu_i, t_k)v(\mu_i, t)]}{\sum_{k=0}^{1} (-1)^k U(\mu_i, t_k) V(\mu_i, t_{1-k})}. \tag{8}$$

For the special case of

$$\delta_0 = 0, \quad f_0(\mathbf{x}) = 0, \quad t_0 = 0 \tag{9a}$$

and if the condition

$$\lim_{t_0 \to 0} \left[\frac{\frac{\mathrm{d}}{\mathrm{d}t} u(\mu_i, t_0)}{\frac{\mathrm{d}}{\mathrm{d}t} v(\mu_i, t_0)} \right] = 0 \tag{9b}$$

is satisfied, the solution (8) reduces to

$$T(\mathbf{x}, t) = \sum_{i=1}^{\infty} \frac{\psi_i(\mathbf{x})}{(\psi_i, \psi_i)} \times (\psi_i, f_1) \frac{u(\mu_i, t)}{\delta_1 u(\mu_i, t_1) + \gamma_1 \dfrac{\mathrm{d}u(\mu_i, t)}{\mathrm{d}t}}. \tag{10}$$

Clearly, several special cases are readily obtainable from the general solutions (8) and (10).

APPLICATION

We now illustrate, with the following examples, the use of the general results given by equations (8) and (10), to obtain solutions for specific heat diffusion problems.

Example 1

Consider a finite rectangular region $0 \le x \le a, 0 \le y \le b$ with boundary at $y = 0$ kept at a prescribed temperature $f(x)$ while the other boundaries are kept at zero temperature. The steady-state temperature $T(x, y)$ satisfies the following system

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\} T(x, y) = 0 \quad \text{in } 0 \le x \le a, \quad 0 \le y \le b \tag{11a}$$

$$T(ka, y) = 0, \qquad k = 0, 1 \tag{11b, c}$$

$$T(x, kb) = (1 - k)f(x), \qquad k = 0, 1. \tag{11d, e}$$

By comparing this problem with the general problem given by equation (1) we write

$$t = y, \quad w(\mathbf{x}) = 1, \quad L_t = -\frac{\partial^2}{\partial y^2}, \quad L = -\frac{\partial^2}{\partial x^2},$$

$$\delta_k = 1, \quad \gamma_k = 0, \quad f_0(\mathbf{x}) = f(x), \quad f_1(\mathbf{x}) = 0,$$
$$B = 1 \quad \text{at } x = ka \quad (k = 0, 1). \tag{12}$$

With various quantities as defined by equation (12), the eigenvalue problem (2) is significantly simplified and the resulting eigenfunctions, eigenvalues and the normalization integral are given respectively by

$$\psi_i(x) = \sin \mu_i x, \quad \mu_i = \frac{\pi}{a} i (i = 1, 2, 3, \ldots),$$

$$N \equiv (\psi_i, \psi_i) = \frac{a}{2} \tag{13}$$

and the elementary solutions of equation (4a) are taken as

$$u(\mu_i, y) = \cosh(\mu_i y), \quad v(\mu_i, y) = \sinh(\mu_i y). \tag{14}$$

Introducing equations (12), (13) and (14) into equation (8), the solution for the problem (11) becomes

$$T(x, y) = \frac{2}{a} \sum_{i=1}^{\infty} \sin(\mu_i x) \frac{\sinh[\mu_i(b - y)]}{\sinh(\mu_i b)} \int_0^a \sin(\mu_i x) f(x) \mathrm{d}x \tag{15}$$

where

$$\mu_i = \frac{\pi}{a} i.$$

Example 2

Along solid cylinder $0 \leq r \leq b$, $0 \leq \phi \leq 2\pi$ is subjected to convective heat transfer at the boundary surface $r = b$ with an environment whose temperature varies around the circumference. The steady-state temperature distribution $T(r, \phi)$ in the cylinder satisfies the following system:

$$\left\{ r \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial \phi^2} \right\} T(r, \phi) = 0$$
$$\text{in } 0 \leq r \leq b, \quad 0 \leq \phi \leq 2\pi \tag{16a}$$

$$\left\{ k \frac{\partial}{\partial r} + h \right\} T(r, \phi) = f(\phi) \quad \text{at } r = b. \tag{16b}$$

A comparison of problem (16) with the general problem (1) reveals that

$$\left.\begin{aligned} & t = r, w(\mathbf{x}) = 1, \quad L_t = -r \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right), \\ & L = -\frac{\partial^2}{\partial \phi^2}, \quad \delta_1 = h, \quad \gamma_1 = k, \quad f_1(\mathbf{x}) = f(\phi). \end{aligned}\right\} \tag{17}$$

With various quantities as defined by equation (17), the eigenvalues of the eigenvalue problem (2) that give a periodic solution with a period 2π are $\mu_i = i$ $(i = 0, 1, 2, \ldots)$. The corresponding two independent eigenfunctions are taken as $\cos(i\phi)$ and $\sin(i\phi)$; and the elementary solutions of equation (4a) become

$$u(\mu_i, r) = r^i, \quad v(\mu_i, r) = r^{-i}. \tag{18}$$

Introducing these results into equation (10), the solution for the problem (16) can be expressed in the form

$$T(r, \phi) = \frac{1}{\pi} \sum_{i=0}^{\infty} \frac{\left(\frac{r}{b} \right)^i}{h + \frac{k}{b} i} \int_0^{2\pi} \cos[i(\phi' - \phi)] f(\phi') \mathrm{d}\phi' \tag{19}$$

where π should be replaced by 2π for $i = 0$. This result is the same as that given in [3], p. 183.

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UNE NOUVELLE RESOLUTION GENERALE DE L'EQUATION DE LA DIFFUSION PERMANENTE DE LA CHALEUR

Résumé — On obtient une solution générale pour l'équation de la diffusion permanente de chaleur par application d'une technique de transformation intégrale. La solution contient une série infinie plus convergente que celle obtenue par la technique dans laquelle toutes les dérivées partielles par rapport aux variables d'espace sont, dans l'équation, traitées par la transformation intégrale. Divers cas spéciaux sont obtenus aisément à partir de la présente solution. L'application à des problèmes spécifiques est illustrée à travers des exemples.

EINE ALTERNATIVE, ALLGEMEINE LÖSUNG DER STATIONÄREN WÄRMELEITUNGSGLEICHUNG

Zusammenfassung—Die stationäre Wärmeleitungsgleichung wurde durch Anwendung der Methode der finiten Integraltransformation in allgemeiner Form gelöst. Die vorliegende Lösung enthält eine unendliche Reihe weniger als diejenige Lösung, die man mittels der Methode der Integraltransformation erhält, bei der alle partiellen Ableitungen nach den Raumkoordinaten mittels Integraltransformation aus der Differentialgleichung entfernt werden. Verschiedene Spezialfälle können aus der vorliegenden Lösung leicht hergeleitet werden. Die Anwendung der allgemeinen Lösung zur Lösung von speziellen Problemen ist anhand von Beispielen erläutert.

К ОБЩЕМУ РЕШЕНИЮ УРАВНЕНИЯ СТАЦИОНАРНОЙ ДИФФУЗИИ ТЕПЛА

Аннотация — Получено общее решение уравнения стационарной диффузии тепла путем использования метода конечных интегральных преобразований. Решение содержит на один ряд меньше бесконечных рядов, чем в случае использования метода интегрального преобразования, когда из дифференциального уравнения исключаются все частные производные по пространственным переменным. Из предлагаемого решения можно легко получить несколько частных случаев. На ряде примеров иллюстрируется применение предлагаемого способа к решению частных задач.